ON THE STABILITY OF SOME CONTINUOUS SYSTEMS SUBJECTED TO RANDOM EXCITATION[1]

by

R. H. Plaut[†] and E. F. Infante[††]

Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, R. I. 02912

[1]This research was supported in part by the Air Force Office of Scientific Research, under Grant No. AFOSR 693-67, in part by the National Aeronautics and Space Adminstration, under Grant No. NGL-40-002-015 and in part by the United States Army, Durham, under Contract No. DA-31-124-ARO-D-270.

[†]Assistant Professor of Applied Mathematics (Research).

[††]Associate Professor of Applied Mathematics.

Abstract

A method for the determination of sufficient conditions for the almost sure stability of some continuous systems of physical interest is presented. The motions of the systems under consideration are assumed to be described by linear partial differential equations with time-varying coefficients of a random nature. The method presented, which is of a rather general form, is restricted for the sake of simplicity and ease of computations and is applied to problems of elastic columns and plates, a cantilever beam subjected to a random follower force, and a string excited by a pressure-type random force. The emphasis both in the computations and in the nature of the method is on simplicity of computations and in the determination of stability conditions with a minimum of assumptions.

Introduction

The stability of systems described by linear ordinary differential equations with stochastic coefficients has been the object of considerable recent interest; in particular, the works of Kozin [1][2], Caughey and Gray [2], Ariaratnam [3], Lepore and Shah [4] and Infante [5] have discussed problems of this nature. The analogous problems for partial differential equations, which naturally arise in the study of the stability of structures subjected to random loads, have usually been reduced to problems of ordinary differential equations by the use of a modal approach in which the amplitude of each mode is governed by an ordinary differential equation and each amplitude is investigated separately [2], [3], [4]. However, the great majority of physically interesting systems are not amenable to such a modal analysis because of the presence of the randomly varying coefficients in the describing partial differential equations. Hence, it is desirable to obtain a method which can be applied directly to these partial differential equations. This is the object of this paper.

It should be pointed out that Wang [6] has considered a similar problem, but his approach differs fundamentally from the one described here; also, his results were obtained by using fundamental properties of semigroups together with the Gronwall inequality, and in the case of ordinary differential equations results obtained in this manner are known to be weak.

---

[2] Numbers in brackets designate References at the end of the paper.

The continuous systems considered in this work are governed by linear partial differential equations with random coefficients; these coefficients are assumed to be stationary and ergodic, in the stochastic case. It is desired to obtain sufficient conditions for the almost sure asymptotic stability of the equilibrium state of the system. The procedure used for this purpose is an extension of the method described in [5] which involves a Liapunov type of approach. The application of this technique to several problems of physical interest, which yield results believed to be new, shows that its simplicity and ease of computation make it an attractive method especially since few assumptions on the nature of the random disturbances are required.

It should be emphasized that the specific techniques used are not necessarily optimal, and that the results can probably be significantly improved. Further work toward this goal seems appropriate.

Statement of the Problem

Consider a continuous system which occupies a bounded domain R in one-, two-, or three-dimensional space $\{\underline{x}\}$, and let C denote the boundary of R. Designate by $w(\underline{x},t)$ the displacement of the system from an equilibrium state which, for simplicity, is taken as $w(\underline{x},t) \equiv 0$; t here represents the time $(t \geq 0)$ and it is assumed that this displacement is governed by a linear partial differential equation of the form

$$\frac{\partial^2 w}{\partial t^2} + 2\xi\frac{\partial w}{\partial t} + \mathscr{L}w + \mathscr{T}(t)w = 0, \ \underline{x} \in R, \ t \geq 0, \tag{1}$$

with homogeneous time-independent boundary conditions of the form

$$\mathscr{B}w = 0, \quad \underline{x} \in C, \tag{2}$$

and initial conditions

$$w(\underline{x},0) = w_o(\underline{x}), \ \frac{\partial w(\underline{x},0)}{\partial t} = v_o(\underline{x}), \quad \underline{x} \in R. \tag{3}$$

In this formulation $\mathscr{B}$, $\mathscr{L}$ and $\mathscr{T}(t)$ are linear spatial differential operators and ξ is a positive constant. In the equation of motion, Eq. (1), the spatial operator terms have been separated, without loss of generality, into the two parts $\mathscr{L}$ and $\mathscr{T}(t)$, where $\mathscr{T}(t)$ includes all the terms with time-varying coefficients and those terms which are not self-adjoint, whereas $\mathscr{L}$ contains only self-adjoint terms with constant coefficients. Hence, whenever w_1 and w_2 satisfy the boundary conditions (2),

$$\int_R w_1 \mathscr{L} w_2 d\underline{x} = \int_R w_2 \mathscr{L} w_1 d\underline{x}. \tag{4}$$

The operator $\mathscr{T}(t)$ has therefore the form

$$\mathscr{T}(t) = \sum_{i=1}^{N} c_i \mathscr{T}_i + \sum_{i=N+1}^{M} f_i(t) \mathscr{T}_i, \tag{5}$$

where the $\mathscr{T}_i$ are time-invariant linear operators, the c_i are constants and the functions $f_i(t)$ are measurable, strictly stationary functions which satisfy an ergodic property ensuring the equality of time and ensemble averages. Under these conditions, if G is a measurable, integrable function defined on the $f_i(t)$ then the limit

$$E\{G[f_i(t)]\} = \lim_{t \to \infty} \frac{1}{t} \int_0^t G[f_i(\tau)] d\tau \tag{6}$$

exists with probability one. It is also assumed that, with the notation $v \equiv \frac{\partial w}{\partial t}$, the problem defined by Eqs. (1), (2) is well posed with respect to the function space whose norm is

$$\rho[w,v] = \{\int_R [w\mathscr{L}w + w^2 + v^2] d\underline{x}\}^{1/2}. \tag{7}$$

It is clear that the formulation of the problem, as well as the choice of the space defined by (7), are motivated by the nature of the physical problems we wish to consider. Equations (1), (2) generalize the partial differential equations of elastic structures and Eq. (7) is intimately related to the concept of energy for such structures.

It is desired to derive sufficient conditions for the almost sure asymptotic stability in the large of the equilibrium state $w(\underline{x},t) \equiv 0$. That is, conditions on the $f_i(t)$ are sought such that the solutions $w(\underline{x},t)$ of (1), (2) with arbitrary initial conditions (3) will satisfy

$$\lim_{t \to \infty} \rho[w(\underline{x},t),\ v(\underline{x},t)] = 0 \tag{8}$$

with probability one. It is evident that if the functions $f_i(t)$ are deterministic and satisfy condition (6) then the sought conditions will imply asymptotic stability in the sense of Liapunov.

Stability Analysis

Consider the functional

$$V[w,v] = \int_R [w\mathscr{L}w + v^2 + bvw + cw^2]d\underline{x}, \tag{9}$$

where b and c are parameters depending on ξ, to be subsequently determined. It is noted that $V[0,0] = 0$, and it is assumed that for all w satisfying the boundary conditions (2)

$$\int_R w\mathscr{L}wd\underline{x} \geq k \int_R w^2 d\underline{x} \tag{10}$$

for some constant $k \geq 0$. Hence

$$V[w,v] \geq \int_R [v^2 + bvw + (k+c)w^2]d\underline{x}, \tag{11}$$

and if b and c are chosen so that

$$b^2 \leq 4(k+c) - \delta \tag{12}$$

for some $\delta > 0$ then V is positive definite and

$$V[w,v] \geq \beta\rho^2[w,v] \tag{13}$$

for some positive constant β.

Let us consider the time rate of change of the functional (9), which is denoted by $\dot{V}[w,v]$,

$$\dot{V}[w,v] = \int_R [v\mathscr{L}w + w\mathscr{L}v + 2v\frac{\partial v}{\partial t} + bv^2 + bw\frac{\partial v}{\partial t} + 2cwv]d\underline{x}. \tag{14}$$

Substitution in this equation of $\frac{\partial v}{\partial t}$ from Eq. (1) yields the time rate of change of the functional (9) along the solutions of (1) as

$$\dot{V}[w,v,t] = \int_R \{v\mathscr{L}w + w\mathscr{L}v + bv^2 + 2cwv + (2v + bw)[-2\xi v - \mathscr{L}w - \mathscr{T}(t)w]\}d\underline{x}. \tag{15}$$

The use of (4) with $w_1 = w$ and $w_2 = v$ allows for the simplification of this expression to

$$\dot{V}[w,v,t] = -\int_R \{bw\mathscr{L}w + (4\xi-b)v^2 + 2(b\xi-c)vw + (2v+bw)\mathscr{T}(t)w\}d\underline{x}. \tag{16}$$

For w and v not both zero, consider the ratio $\dot{V}/V$ with $\dot{V}$ given by Eq. (16) and V by Eq. (9). Let a scalar function $\lambda_M(t)$ be such that

$$\frac{\dot{V}[w,v,t]}{V[w,v]} \leq \lambda_M(t) \tag{17}$$

for all w and v satisfying the boundary conditions (2). Integration of this expression on $[0,t]$ yields

$$V(t) \leq V(0)e^{t[\frac{1}{t}\int_0^t \lambda_M(\tau)d\tau]}, \tag{18}$$

where $V(t)$ denotes $V[w(\underline{x},t), v(\underline{x},t)]$ and $V(0) = V[w_o(\underline{x}), v_o(\underline{x})]$. Inequality (18) provides an exponential bound on the motion of the system at any time t in terms of the quantity $\frac{1}{t}\int_o^t \lambda_M(\tau)d\tau = \frac{1}{t}\int_o^t G[f_i(\tau)]d\tau$. An example of this type of result, applied to a specific problem, is given in [7]. In the present work, however, let us restrict ourselves to the discussion of the system behavior as $t \to \infty$ and we force the restrictions on the $f_i(t)$ for asymptotic stability to the form of conditions on expectations.

From Equations (6), (13) and (18) it is clear that if

$$E\{\lambda_M(t)\} \leq -\epsilon \tag{19}$$

for some constant $\epsilon > 0$, then with probability one it will follow that $V(t) \to 0$ and hence $\rho[w(\underline{x},t), v(\underline{x},t)] \to 0$ as $t \to \infty$. Hence, Eq. (19) represents a sufficient condition for the almost sure asymptotic stability in the large of $w(\underline{x},t) \equiv 0$.

The application of this general method to a particular example involves the following procedure: First of all, a constant k that satisfies Eq. (10) must be found; then the function $\lambda_M(t)$ of Eq. (17) is determined and the stability conditions expressed by the inequality (19) are forced in the form of the desired expectations, such as $E\{|f_i(t)|\}$ or $E\{f_i^2(t)\}$; finally, the constants b and c which optimize these conditions are determined subject to the constraint expressed by Eq. (12).

It should be noted that the method is very general, except for the choice of the specific functional (9) which has been selected. The reason for this particular choice rests on the fact that experience has indicated this functional to yield good results with only a moderate amount of computations. The selection of the form of the functional which yields optimal results is a further area where research seems indicated, although not too promising.

Some Examples

The procedure outlined above will now be applied to some linear continuous systems subjected to random loads. Two observations are important regarding these examples: i) all the examples are of a nonconservative nature, even if the random functions were replaced by constants and, ii) in none of the examples are the equations of motion separable, hence a modal stability analysis is not feasible.

In the first example the computations and procedures are carried out in detail; in the other two examples, for economy of space, the computations are simply indicated but the missing steps can be easily filled in by the interested reader.

Example 1. Consider the partial differential equation of motion

$$\frac{\partial^2 w}{\partial t^2} + 2\xi\frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} + p(t)\frac{\partial^2 w}{\partial x^2} + u(t)\frac{\partial w}{\partial x} = 0, \quad 0 < x < 1, \quad t \geq 0 \qquad (20)$$

where $w = w(x,t)$ and x is one-dimensional, and the boundary conditions

$$w(0,t) = \frac{\partial^2 w(0,t)}{\partial x^2} = w(1,t) = \frac{\partial^2 w(1,t)}{\partial x^2} = 0, \quad t \geq 0. \qquad (21)$$

It is of interest to note that if $u(t) \equiv \bar{u} =$ constant then Eqs. (20), (21) describe the motion of a "two-dimensional" elastic plate in a supersonic airstream of velocity $\bar{u}$ and subjected to uniform normal forces $p(t)$ at the simply supported edges $x = 0$ and $x = 1$, a configuration illustrated in Figure 1(a). In accordance with piston theory, the terms $2\xi\frac{\partial w}{\partial t} + \bar{u}\frac{\partial w}{\partial x}$ represent the aerodynamic forces due to the airstream. If $u(t) \equiv 0$, then Eqs. (20) and (21) represent a simply supported elastic column under an axial load $p(t)$ with ξ as the coefficient of viscous damping, Figure 1(b). In this latter case, the problem is separable and has been discussed in detail in [7].

For Eqs. (20) and (21) it is clear that

$$\begin{aligned} \mathscr{L}w &= \frac{\partial^4 w}{\partial x^4}, \quad \mathscr{T}(t)w = p(t)\frac{\partial^2 w}{\partial x^2} + u(t)\frac{\partial w}{\partial x}, \\ V[w,v] &= \int_0^1 \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + v^2 + bvw + cw^2\right]dx, \end{aligned} \tag{22}$$

where integration by parts has been used. Application of straightforward inequalities or, even more fundamentally, of a variational technique with Lagrange multipliers yields the value of $k = \pi^4$ satisfying Eq. (10) and a function $\lambda_M(t)$ satisfying Eq. (17). Let us illustrate this procedure in the determination of $\lambda_M(t)$.

In this case, consider the functional

$$\begin{aligned} \mathring{V} - \lambda V = -\int_0^1 \Big\{ &(\lambda+b)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + (\lambda+4\xi-b)v^2 + (\lambda b+2b\xi-2c)vw + \\ &+ p(t)(2v+bw)\frac{\partial^2 w}{\partial x^2} + \lambda cw^2 + 2u(t)v\frac{\partial w}{\partial x}\Big\}dx. \end{aligned} \tag{23}$$

The variation of this functional is written in the form

$$\delta(\dot{V}-\lambda V) = -2\int_0^1 \{\phi(w,v,t)\delta v + \psi(w,v,t)\delta w\}dx, \tag{24}$$

where

$$\phi(w,v,t) = (\lambda+4\xi-b)v + \frac{1}{2}(\lambda b+2b\xi-2c)w + + u(t)\frac{\partial w}{\partial x} + p(t)\frac{\partial^2 w}{\partial x^2} \tag{25}$$

and

$$\psi(w,v,t) = \frac{1}{2}(\lambda b+2b\xi-2c)v - u(t)\frac{\partial v}{\partial x} + p(t)\frac{\partial^2 v}{\partial x^2} + + \lambda cw + bp(t)\frac{\partial^2 w}{\partial x^2} + (\lambda+b)\frac{\partial^4 w}{\partial x^4} . \tag{26}$$

Forcing the variation (24) to vanish yields the two equations $\phi = 0$ and $\psi = 0$, whose linearity allows for the elimination of one of the unknowns, say v, yielding the fourth order equation in w

$$[(\lambda+b)(\lambda+4\xi-b) - p^2(t)]\frac{\partial^4 w}{\partial x^4} + [(2c+2\xi b - b^2)p(t) + u^2(t)]\frac{\partial^2 w}{\partial x^2} + + [\lambda c(\lambda+4\xi-b) - (b\xi-c+\frac{\lambda b}{2})^2]w = 0. \tag{27}$$

This ordinary differential equation, for $p(t)$ and $u(t)$ fixed quantities, with the boundary conditions (21) yields the eigenfunctions $w(x) = \sin n\pi x$ and the eigenvalues $\lambda = \lambda_n(t)$ of the form

$$\lambda_n(t) = -2\xi \pm \sqrt{\frac{n^4\pi^4 p^2(t) + \alpha_n p(t) + \beta_n + n^2\pi^2 u^2(t)}{n^4\pi^4 + c - b^2/4}}, \tag{28}$$

$$n = 1, 2, \ldots,$$

where $\alpha_n = n^2\pi^2(2c+2\xi b-b^2)$, $\beta_n = n^4\pi^4(b-2\xi)^2 + 2\xi(2\xi-b)c + c^2$. Setting $\lambda_M(t) = \max\limits_{n=1,2,\ldots} \lambda_n(t)$ yields the desired function. This maximization is somewhat complicated, and for computational simplicity it is desirable that the maximum occur for a fixed value of n, independently of the values of $u(t)$ and $p(t)$. This is easily accomplished by letting $\alpha_n = 0$, i.e. forcing

$$c = b(b-2\xi)/2; \tag{29}$$

it is noted that the functional $V[w,v]$ is such that this simplifying condition can be satisfied along with condition (12), which in this case takes the form

$$b^2 \leqq 4(\pi^4 + c) - \delta, \tag{30}$$

by an appropriate choice of the parameters b and c.

With the use of condition (29) it is clear that the maximum of (28) occurs for $n = 1$ if $0 \leqq b \leqq 4\xi$, and therefore

$$\lambda_M(t) = -2\xi + \sqrt{\frac{\pi^4 p^2(t) + \pi^2 u^2(t)}{\pi^4 - \xi b + b^2/4} + (b-2\xi)^2}, \quad 0 \leq b \leq 4\xi. \tag{31}$$

Hence, the stability condition (19) becomes, in this case,

$$E\left\{\sqrt{\frac{\pi^4 p^2(t) + \pi^2 u^2(t)}{\pi^4 - \xi b + b^2/4} + (b-2\xi)^2}\right\} \leq 2\xi - \epsilon. \tag{32}$$

Application of Schwarz's inequality then immediately yields

$$\pi^4 E\{p^2(t)\} + \pi^2 E\{u^2(t)\} \leq [4\xi^2 - (b-2\xi)^2][\pi^4 - \xi b + b^2/4] - \epsilon, \tag{33}$$

and the optimal choice for the parameter b consistent with (30) and with $0 \leq b \leq 4\xi$ is immediately found as

$$b = \begin{cases} 2\xi & \text{for } \xi \leq \pi^2/\sqrt{2} \\ 2\xi + \sqrt{2(2\xi^2 - \pi^4)} & \text{for } \xi \geq \pi^2/\sqrt{2} \end{cases}. \tag{34}$$

Hence, with this value of b, the stability condition becomes

$$E\{p^2(t)\} + \frac{1}{\pi^2}E\{u^2(t)\} \leq \begin{cases} 4\xi^2(1 - \xi^2/\pi^4) - \epsilon, & \xi \leq \pi^2/\sqrt{2} \\ \pi^4 - \epsilon, & \xi \geq \pi^2/\sqrt{2} \end{cases}. \tag{35}$$

Therefore, if $p(t)$ and $u(t)$ are such that Eq. (35) is satisfied then the system governed by Eqs. (20) and (21) is almost surely asymptotically stable; the stability results given by Eq. (35) are depicted in graphical form in Figure 2.

It should be noted that the stability criterion (35) is applicable to every set of functions $p(t)$ and $u(t)$ for which $E\{p^2(t)\}$ and $E\{u^2(t)\}$ are defined. Hence, if we consider the special case $p(t) \equiv p_o$ = constant and $u(t) \equiv 0$ which describes the case of the column in Figure 1(b) under a constant load p_o, we note that the buckling load is then given by $p_o = \pi^2$ for all ξ; therefore, if $E\{p^2(t)\} > \pi^4$ we are assured that there exists at least one function $p(t)$ in this class that produces instability, and hence π^4 is an upper bound for the stability region of the type shown in Figure 2. This remark shows that the result obtained, which is only a sufficient condition for stability, is relatively sharp.

Stability criteria in terms of the expectations $E\{|p(t)|\}$ and $E\{|u(t)|\}$ are also easily obtained from Eq. (32). Since $E\{|\nu+\mu|\} \leq E\{|\nu|\} + E\{|\mu|\}$ and $\sqrt{\nu^2} = |\nu|$, a sufficient condition for (32) to hold is

$$\pi^2 E\{|p(t)|\} + \pi E\{|u(t)|\} \leq (2\xi - |b-2\xi|)\sqrt{\pi^4 - \xi b + b^2/4} - \epsilon. \qquad (36)$$

The optimal choice of b for this inequality is given by

$$b = \begin{cases} 2\xi & \text{for } \xi \leqq .96\pi^2 \\ (5\xi + \sqrt{9\xi^2 - 8\pi^4})/2 & \text{for } \xi > .96\pi^2 \end{cases} \tag{37}$$

which yields the stability domain depicted in Figure 3.

Example 2. As a second example, let us consider the stability of a cantilevered column subjected at its free end to a random follower force, as indicated in Figure 4. Viscous damping is assumed, and therefore the equation of motion for the lateral displacement $w(x,t)$ is given by

$$\frac{\partial^2 w}{\partial t^2} + 2\xi\frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} + p(t)\frac{\partial^2 w}{\partial x^2} = 0, \quad 0 < x < 1, \quad t \geqq 0, \tag{38}$$

with boundary conditions

$$w(0,t) = \frac{\partial w(0,t)}{\partial x} = \frac{\partial^2 w(1,t)}{\partial x^2} = \frac{\partial^3 w(1,t)}{\partial x^3} = 0, \; t \geqq 0. \tag{39}$$

The same procedure as the one given in detail for Example 1 is followed for the determination of $\lambda_M(t)$. Forcing the stability conditions to be in the form of restrictions on $E\{p^2(t)\}$, the optimal value of b is found as

$$b = \begin{cases} 2\xi & \text{for} \quad \xi \leq \pi^2/4 \\ 2\xi + \sqrt{4\xi^2 - \pi^4/4} & \text{for} \quad \xi > \pi^2/4 \end{cases} \tag{40}$$

leading to the stability criterion

$$E\{p^2(t)\} \leq \begin{cases} 4\xi^2 - 32\xi^4/\pi^4 - \epsilon & \text{if} \quad \xi \leq \pi^2/4 \\ \pi^4/8 - \epsilon & \text{if} \quad \xi \geq \pi^2/4 \end{cases} . \tag{41}$$

The stability region defined by these equations is depicted in Figure 5.

It is of interest to recall the stability criterion for the case of a constant follower force, $p(t) \equiv \bar{p} = \text{constant}$. For $\xi = 0$, Beck [8] has shown that the column is stable (but not asymptotically stable) if and only if

$$\bar{p} \leq 2.03\pi^2 - \epsilon, \tag{42}$$

and it is known that for $\xi > 0$ this condition is sufficient for asymptotic stability [9]. A comparison of the result obtained above with this condition suggests that the stability criterion (41) is probably not a very sharp one, and that it can probably be improved. However restrictive this condition, it must be pointed out that it was obtained in a simple and straightforward manner; it is believed to be the first stability criterion obtained for a cantilever subjected to a time-dependent follower force.

Example 3. As a final example, consider the equation

$$\frac{\partial^2 w}{\partial t^2} + 2\xi\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} + f(t)\frac{\partial w}{\partial x} = 0, \quad 0 < x < 1, \quad t \geq 0, \tag{43}$$

with the associated boundary conditions

$$w(0,t) = w(1,t) = 0, \quad t \geq 0. \tag{44}$$

Here $w(x,t)$ represents the lateral displacement of a string stretched between fixed ends and subjected to a transverse load $f(t)\frac{\partial w}{\partial x}$ and damping force $2\xi\frac{\partial w}{\partial t}$. The functional $V[w,v]$ then has the form

$$V[w,v] = \int_0^1 [(\frac{\partial w}{\partial x})^2 + v^2 + bvw + cw^2]dx. \tag{45}$$

Following again the procedure of Example 1 yields the optimal values of the parameters b and c to be

$$b = \frac{2\pi^2\xi}{\pi^2 + \xi^2}, \quad c = 0, \tag{46}$$

upon which the stability criterion in terms of $E\{f^2(t)\}$ is obtained immediately as

$$E\{f^2(t)\} \leqq \frac{4\pi^2\xi^2}{\pi^2 + \xi^2} - \epsilon, \tag{47}$$

a criterion depicted graphically in Figure 6.

Concluding Remarks

As the three above examples illustrate, the method suggested in this paper is a simple, straightforward tool for the determination of sufficient conditions for the almost sure asymptotic stability of some continuous systems subjected to random excitation. This method is not an approximate one and it does not require that the equations of motion be separable, as the modal approach demands. Furthermore, the techniques used are relatively independent of the non-self-adjointness of the equations and the stability criteria obtained can be put in the form of simple equations involving the expectations of the disturbances.

The computations required are rather elementary, involving only the use of some rather well-known inequalities or, at most, the solution of simple eigenvalue problems.

The stability criteria obtained are only sufficient, as is to be expected; the choice of a definite form for the functional V allows for the ease of the computations but is, on the other hand, a restriction which may cause the criteria obtained to be very conservative. The optimal choice of this functional remains an open problem.

References

1. Kozin, F., "On Almost Sure Stability of Linear Systems with Random Coefficients", Journal of Mathematics and Physics, Vol. 42, 1963, pp. 59-67.

2. Caughcy, T. K., and Gray, A. H., Jr., "On the Almost Sure Stability of Linear Dynamic Systems with Stochastic Coefficients", Journal of Applied Mechanics, Vol. 32, No. 2, Trans. ASME, Vol. 87, Series E, June 1965, pp. 365-372.

3. Ariaratnam, S. T., "Dynamic Stability of a Column Under Random Loading", Dynamic Stability of Structures, Proceedings of International Conference, Pergamon Press, New York, 1967, pp. 267-284.

4. Lepore, J. A., and Shah, H. C., "Dynamic Stability of Axially Loaded Columns Subjected to Stochastic Excitations", AIAA Journal, Vol. 6, No. 8, August 1968, pp. 1515-1521.

5. Infante, E. F., "On the Stability of Some Linear Non-autonomous Random Systems", Journal of Applied Mechanics, Vol. 35, No. 1, Trans. ASME, Vol. 90, Series E, March 1968, pp. 7-12.

6. Wang, P. K. C., "On the Almost Sure Stability of Linear Stochastic Distributed-Parameter Dynamical Systems", Journal of Applied Mechanics, Vol. 33, No. 1, Trans. ASME, Vol. 88, Series E, March 1966, pp. 182-186.

7. Infante, E. F., and Plaut, R. H., "On the Stability of a Column Subjected to a Time-Dependent Axial Load", to appear in AIAA Journal.

8. Beck, M., "Die Knicklast des einseitig eingespannten, tangential gedrückten Stabes", <u>Zeitschrift für angewandte Mathematik und Physik</u>, Vol. 3, 1952, pp. 225-228 and 476-477 (errata).

9. Leipholz, H., "Über den Einfluss der Dämpfung bei nichtkonservativen Stabilitätsproblemen elastischer Stäbe", <u>Ingenieur-Archiv</u>, Vol. 33, 1964, pp. 308-321.

Captions for Figures

Fig. 1	Systems in Example 1
Fig. 2	Stability Region for Example 1
Fig. 3	Stability Region for Example 1
Fig. 4	System in Example 2
Fig. 5	Stability Region for Example 2
Fig. 6	Stability Region for Example 3

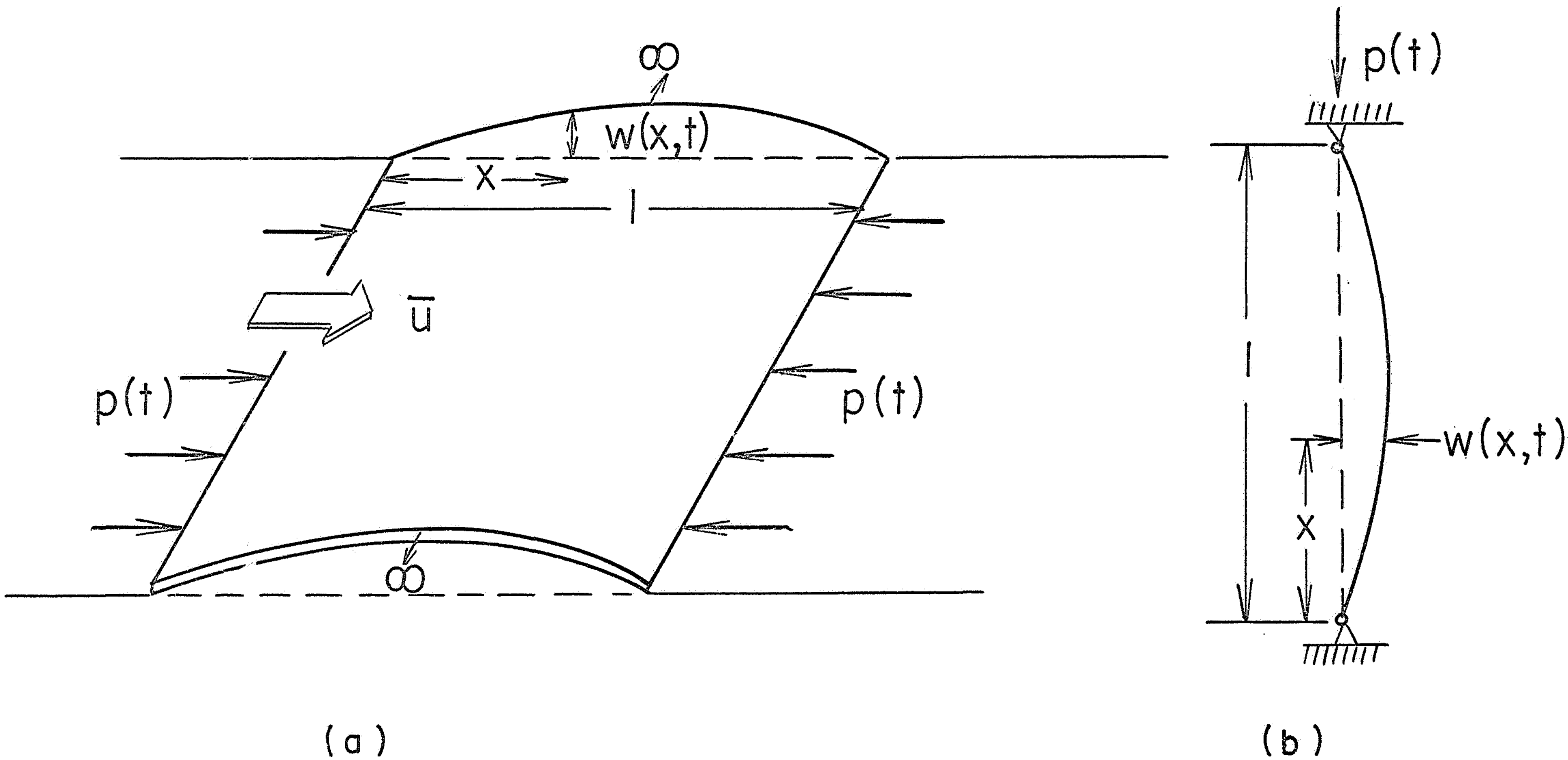

FIG. 1

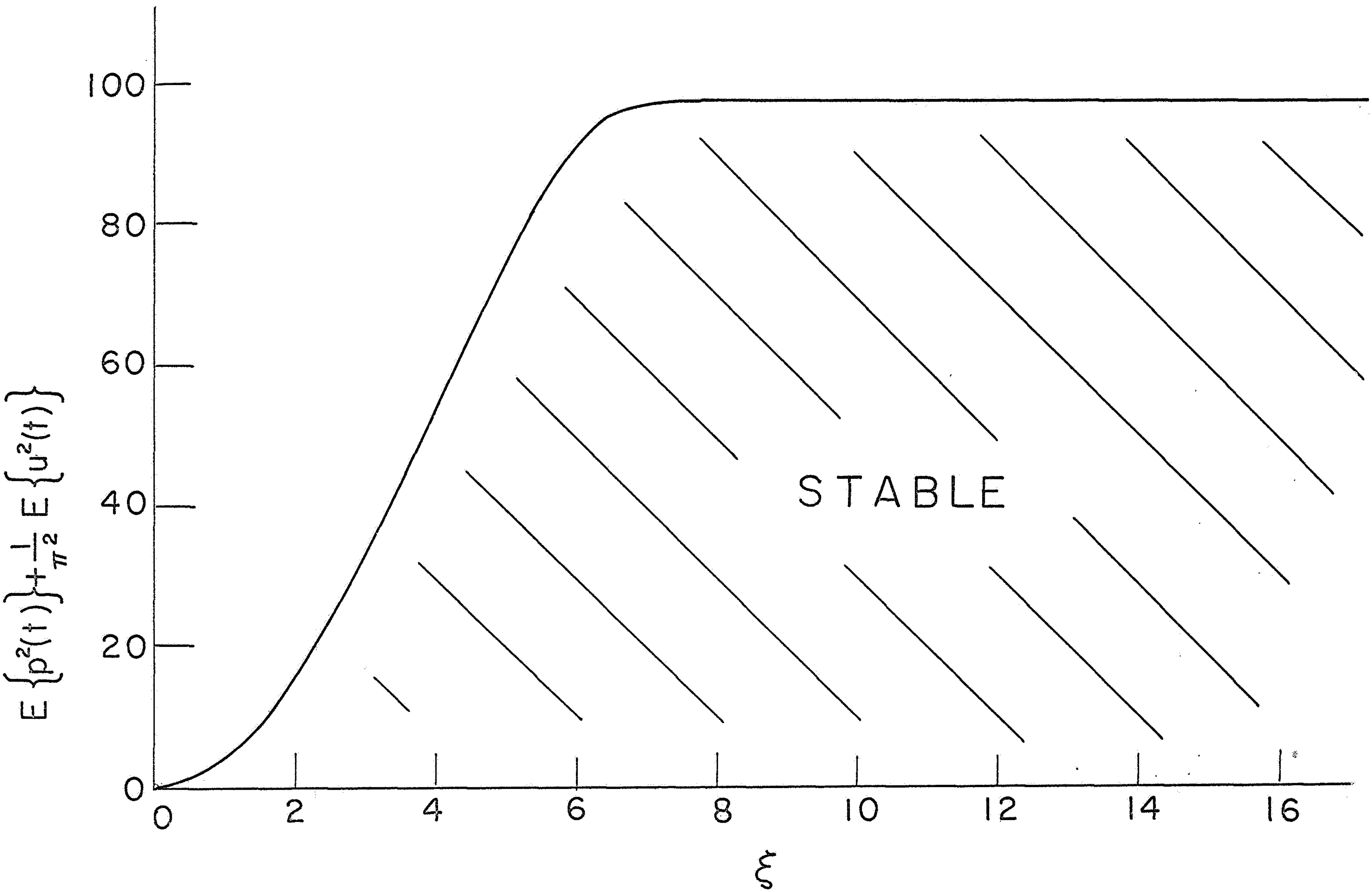
$E\left\{p^2(t)\right\}+\frac{1}{\pi^2}E\left\{u^2(t)\right\}$
100
80
60
40
20
0
0
2
4
6
8
10
12
14
16
ξ
STABLE

FIG. 2

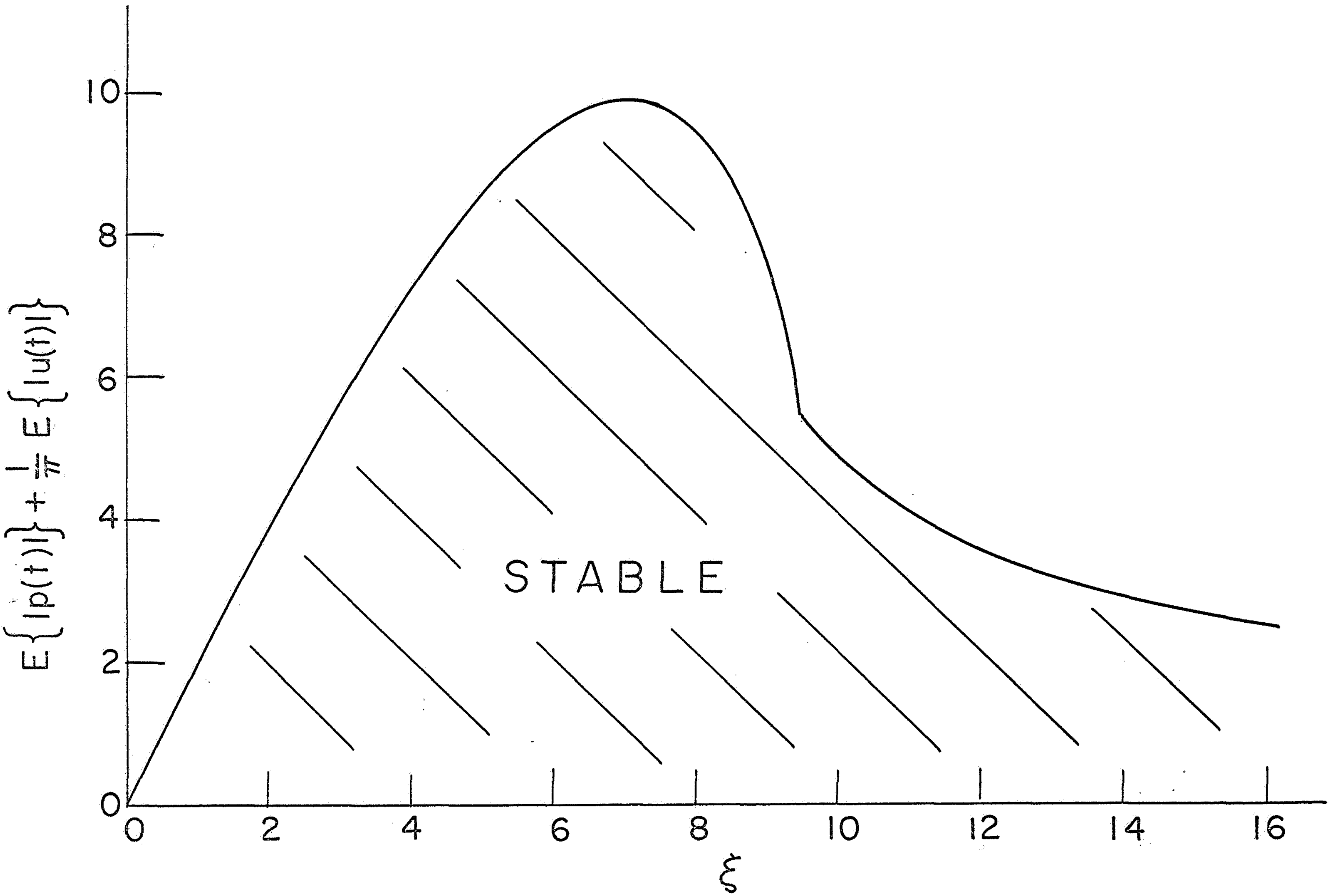

FIG. 3

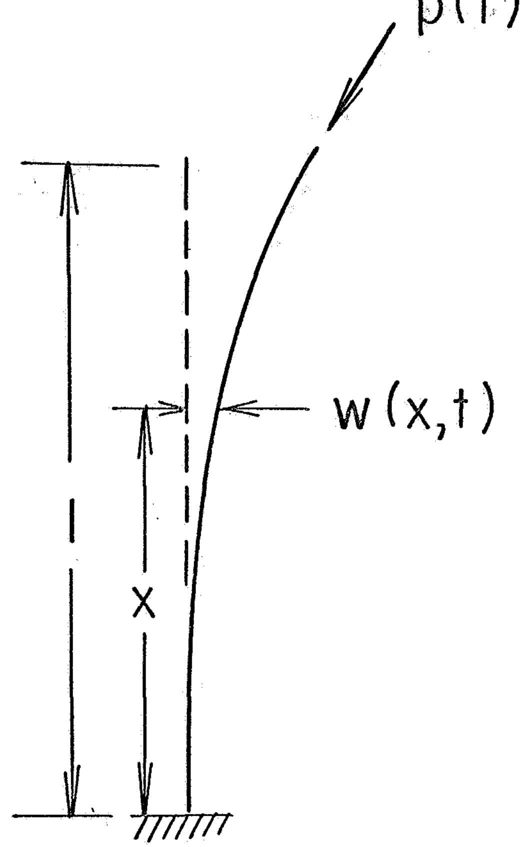

FIG. 4

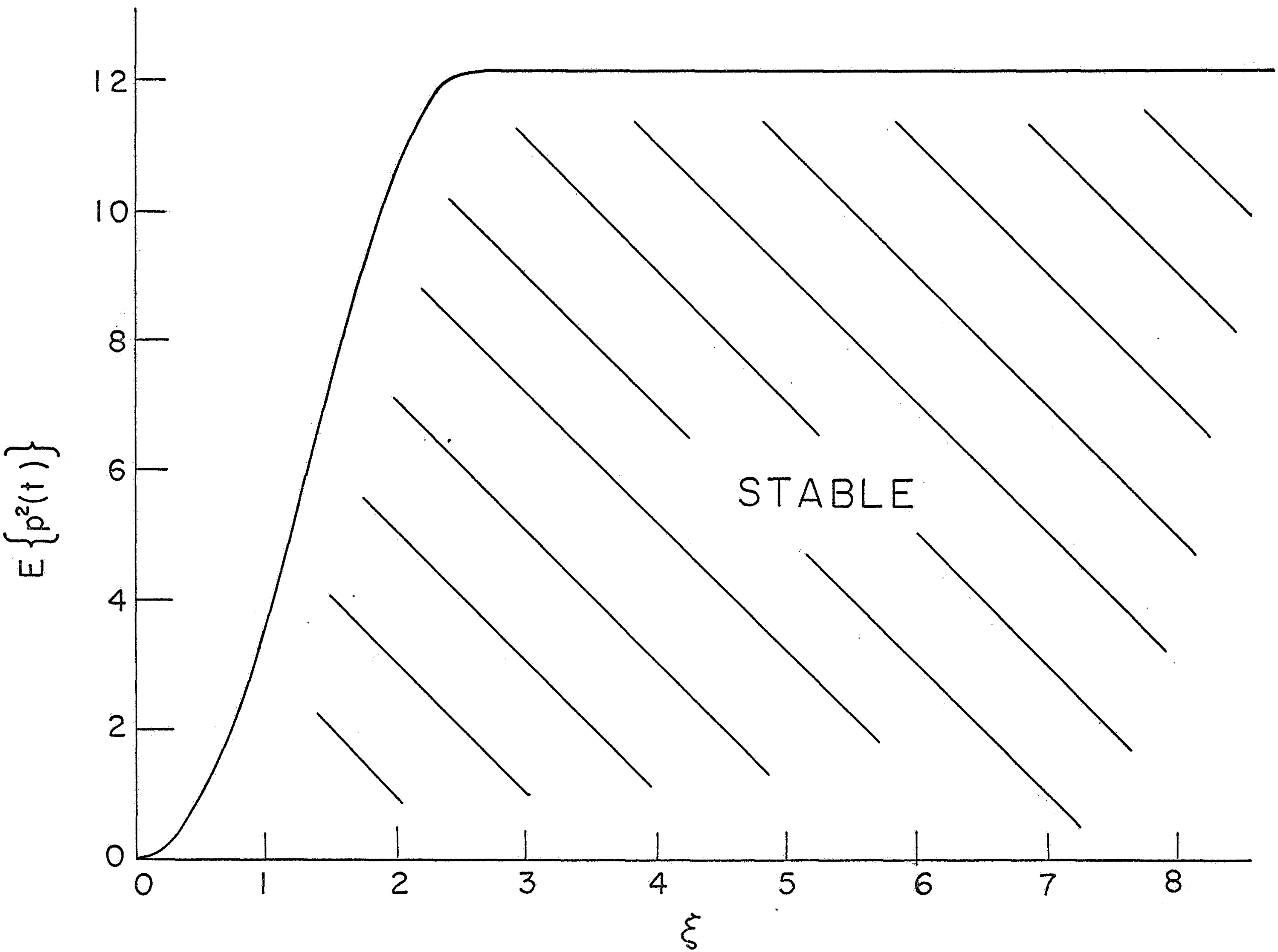

FIG. 5

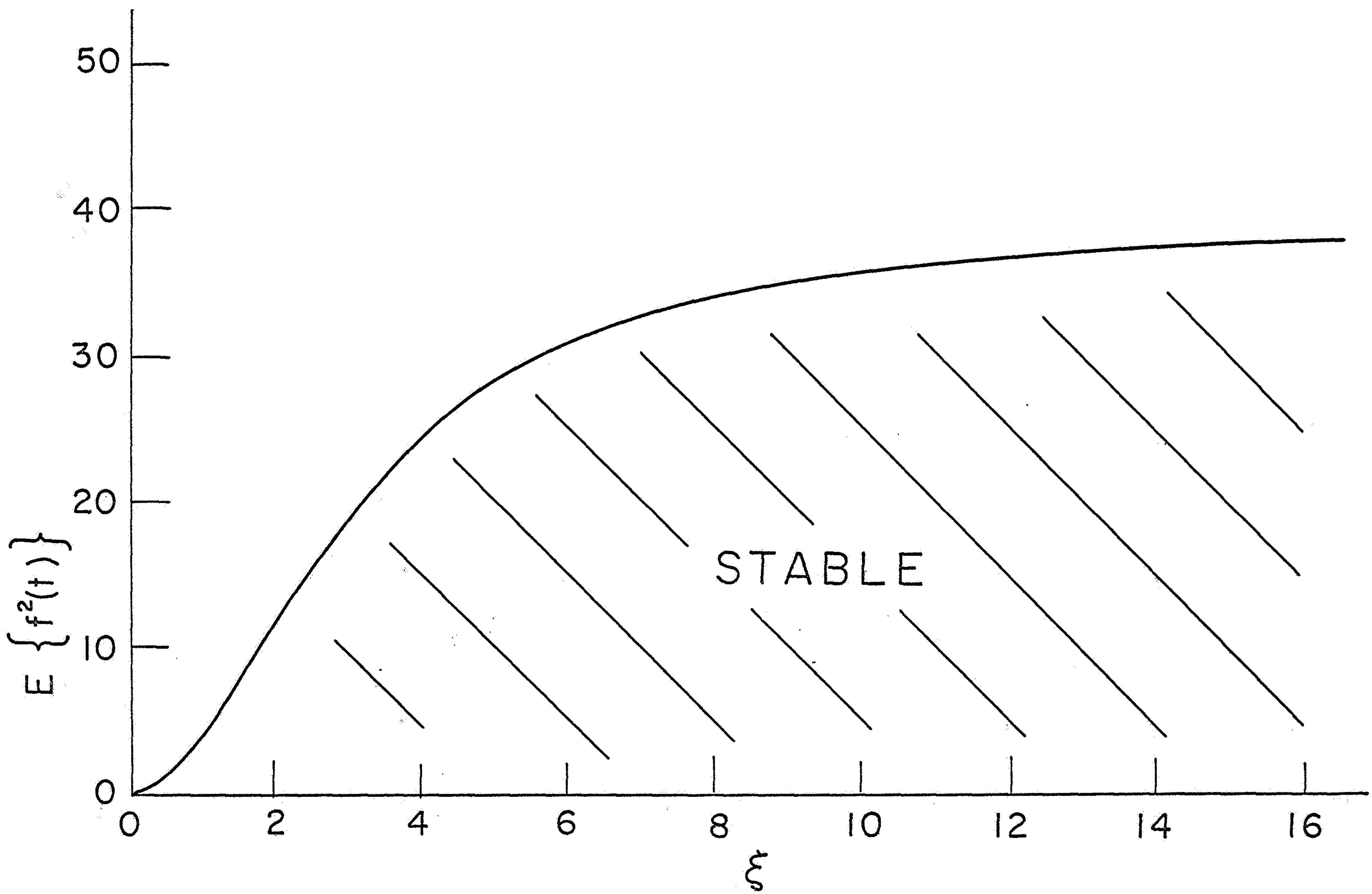

FIG. 6